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TIME-OPTIMAL CONTROL IN A THIRD-ORDER SYSTEM†

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A third-order linear controlled system which simulates the motion of an inertial object acted upon by a control force with a bounded rate of change is considered. The time-optimal open-loop control of the system is constructed. The feedback optimal control is given in closed form. © 1998 Elsevier Science Ltd. All rights reserved.

1. FORMULATION OF THE PROBLEM

We consider a system with a single degree of freedom, described by the equations

$$\dot{x}_1 = x_2, \quad m\dot{x}_2 = F \tag{1.1}$$

where x_1 is a generalized coordinate, x_2 is a generalized velocity, m is a constant inertial characteristic (the mass or moment of inertia), F is the control (the force or the moment of the force) and dots denote derivatives with respect to the time t.

When formulating optimal control problems, it is usually assumed that the absolute magnitude of the force F is bounded by a constant F_0, that is, $|F| \leqslant F_0$. In the case of a time-optimal control problem it is well known [1] that this constraint leads to the bang–bang form of optimal control. In this case, the force $F(t)$ takes limiting values $\pm F_0$ and instantaneously switches from one of these values to the other. Such a control is not always practicable, for example, when an electric drive is used to realize the control.

In this paper, we assume that there is a more realistic constraint on the rate of change of the control force of the form

$$|\dot{F}| \leqslant v_0 \tag{1.2}$$

where $v_0 > 0$ is a specified constant. We shall also assume that the bound on the absolute magnitude of the force is not attained and $|F(t)| < F_0$ always.

Making the change of variables

$$x_1 = (v_0/m)x, \quad x_2 = (v_0/m)y, \quad F = v_0 z$$

we reduce (1.1) and constraint (1.2) to the form

$$\dot{x} = y, \quad \dot{y} = z, \quad \dot{z} = u, \quad |u| \leqslant 1 \tag{1.3}$$

Here, the variables x, y and z are phase coordinates and u plays the role of a bounded control.

The initial conditions for system (1.3) are specified in the form

$$x(0) = x_0, \quad y(0) = y_0, \quad z(0) = z_0 \tag{1.4}$$

where the initial instant of time is assumed to be equal to zero without any loss in generality.

We now formulate the problem of constructing a control $u(t)$ which satisfies the constraint $|u(t)| \leqslant 1$ when $t \geqslant 0$ and which transfers system (1.3) from an arbitrary initial state (1.4) to a specified terminal manifold

$$x(T) = 0, \quad y(T) = 0 \tag{1.5}$$

for arbitrary $z(T)$ after the shortest time T.

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In addition to determining the open-loop control, the problem of the feedback time-optimal control for system (1.3) will also be solved. This control $u(x, y, z)$, which is expressed as a function of the current (or initial) phase coordinates x, y, z, ensures that system (1.3) is brought to the specified terminal manifold (1.5) after the shortest time.

2. THE MAXIMUM PRINCIPLE

We now apply the maximum principle [1] to the time-optimal control problem (1.3)–(1.5). We set up the Hamiltonian function

$$H = p_x y + p_y z + p_z u \tag{2.1}$$

and write down the conjugate equations

$$\dot{p}_x = 0, \quad \dot{p}_y = -p_x, \quad \dot{p}_z = -p_y \tag{2.2}$$

Here p_x, p_y, p_z are the conjugate variables. System (2.2) is integrated subject to the transversality condition $p_z(T) = 0$ which corresponds to the condition that $z(T)$ is not fixed, and we obtain

$$p_x = c_x, \quad p_y = c_y + c_x\tau, \quad p_z = c_y\tau + c_x\tau^2/2 \tag{2.3}$$

Here, $\tau = T - t$ is the time measured from the end of the process (the "inverse" time) and c_x and c_y are arbitrary constants. The condition for the Hamiltonian (2.1) to be a maximum with respect to u subject to the constraint $|u| \leqslant 1$ from (1.3) gives $u(t) = \operatorname{sign} p_z(t)$. It follows from formula (2.3) for p_z that the function $p_z(t)$ changes sign not more than once when $t \leqslant T$, $\tau \geqslant 0$. Consequently, the optimal control $u(t) = \pm 1$ has not more than one switching when $t \leqslant T$.

3. OPEN-LOOP CONTROL

We denote the lengths of the two possible segments of constancy of the control $u(t)$ by θ_1 and θ_2 and the value of $u(t)$ in the first of these segments by $\sigma = \pm 1$. The optimal control can then be represented in the form

$$\begin{aligned} &u(t) = \sigma \ \text{ when } t \in (0, \theta_1) \\ &u(t) = -\sigma \ \text{ when } t \in (\theta_1, T), \ \theta_1 + \theta_2 = T \end{aligned} \tag{3.1}$$

We now substitute control (3.1) into system (1.3) and integrate it subject to the initial conditions (1.4). We obtain

$$\begin{aligned} &x(t) = x_0 + y_0 t + z_0 t^2/2 + \sigma t^3/6 \\ &y(t) = y_0 + z_0 t + \sigma t^2/2, \quad z(t) = z_0 + \sigma t \ \text{ when } t \in (0, \theta_1) \\ &x(t) = x_0 + y_0\theta_1 + z_0\theta_1^2/2 + \sigma\theta_1^3/6 + (y_0 + z_0\theta_1 + \sigma\theta_1^2/2)(t - \theta_1) + \\ &+(z_0 + \sigma\theta_1)(t - \theta_1)^2/2 - \sigma(t - \theta_1)^3/6 \\ &y(t) = y_0 + z_0\theta_1 + \sigma\theta_1^2/2 + (z_0 + \sigma\theta_1)(t - \theta_1) - \sigma(t - \theta_1)^2/2 \\ &z(t) = z_0 + \sigma\theta_1 - \sigma(t - \theta_1) \ \text{ when } t \in (\theta_1, T) \end{aligned} \tag{3.2}$$

Substituting solution (3.2) into condition (1.5), we obtain two relations and, on solving these for x_0 and y_0, we obtain

$$\begin{aligned} &x_0 = z_0 T^2/2 + \sigma(\theta_1^3 + 3\theta_1^2\theta_2 - \theta_2^3)/3 \\ &y_0 = -z_0 T - \sigma(\theta_1^2 + 2\theta_1\theta_2 - \theta_2^2)/2 \end{aligned} \tag{3.3}$$

The following notation is introduced

$$\xi = z_0^{-3}x_0, \quad \eta = z_0^{-1}|z_0|^{-1}y_0, \quad \zeta = \operatorname{sign} z_0$$
$$s = |z_0|^{-1}T, \quad \lambda = \theta_2 T^{-1} \quad (z_0 \neq 0) \tag{3.4}$$
$$X(\lambda) = (1 - 3\lambda^2 + \lambda^3)/3, \quad Y(\lambda) = \lambda^2 - 1/2$$

Relations (3.3) then takes the form

$$\zeta(\xi s^{-3} - s^{-1}/2) = \sigma X(\lambda), \quad \zeta(\eta s^{-2} + s^{-1}) = \sigma Y(\lambda) \tag{3.5}$$

When $z_0 = 0$, relations (3.3) give

$$x_0 T^{-3} = \sigma X(\lambda), \quad y_0 T^{-2} = \sigma Y(\lambda) \tag{3.6}$$

When the parameter λ changes from 0 to 1, a point with coordinates $X(\lambda)$, $Y(\lambda)$ traverses the arc of the curve which joins points A_1 and A_2 with coordinates (1/3, –1/2), (–1/3, 1/2). When $\lambda \in [0, 1]$ and $\sigma = \pm 1$, points with coordinates $\sigma X(\lambda)$, $\sigma Y(\lambda)$ form a closed curve Γ which is symmetric about the origin of coordinates and has corner points A_1 and A_2 (see Fig. 1). The curve Γ bounds a convex domain containing the origin of the system of coordinates.

The solution of the time-optimal open-loop control problem (1.3)–(1.5) can then be represented as follows.

We initially assume that $z_0 \neq 0$ and determine ξ, η, ζ in accordance with (3.4) from (1.5) using the specified initial data x_0, y_0, z_0. The left-hand sides of relations (3.5) specify the coordinates of a certain point P which depends on the parameter $s \in [0, \infty)$. As s changes from ∞ to 0, P moves along a smooth semi-infinite curve from the origin of the system of coordinates (when $s \to \infty$) to infinity (when $s \to 0$). This point falls at least once on the closed curve Γ which encircles the origin of the system of coordinates. The least value of $s = s_*$ for which $P \in \Gamma$ is found numerically. According to (3.4), the optimal time is equal to $T = |z_0|s_*$. The position of the point P on the curve Γ, when $s = s_*$, determines the values of the parameters $\sigma = \pm 1$ and $\lambda \in [0, 1]$. By virtue of (3.4), the lengths of the segments of constancy of the control are equal to $\theta_1 = (1 - \lambda)T$ and $\theta_2 = \lambda T$.

When $z_0 = 0$, we consider equalities (3.6) instead of (3.5). The left-hand sides of these equalities specify the coordinates of the point P which depends on the parameter T. When T changes from ∞ to 0, the point P moves along a semicubic parabola from the origin of the system of coordinates (when $T \to \infty$) to infinity (when $T \to 0$). The least value of the parameter T for which $P \in \Gamma$ is the optimal time. The values of the parameters σ, λ, θ_1, θ_2 are determined from the position of the point P on Γ, as in the case when $z_0 \neq 0$.

When the quantities σ, θ_1, θ_2 have been determined, the optimal control $u(t)$ and the corresponding optimal trajectory are specified by equalities (3.1) and (3.2). The proposed algorithm completely

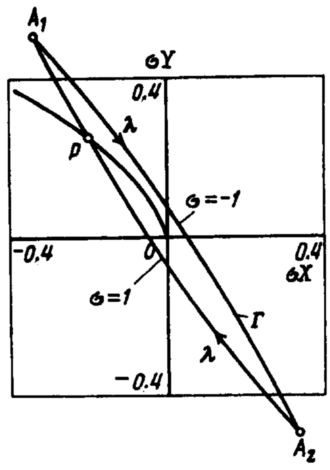

Fig. 1.

determines the solution of the time-optimal open-loop control problem. According to the construction, this solution is unique.

As an example, we present the results of the determination of the optimal control for the initial data

$$x_0 = -72 + 27\sqrt{3} \approx -25.2, \quad y_0 = 3, \quad z_0 = 1$$

In this case, one obtains

$$T = s = 6, \quad \sigma = 1, \quad \theta_1 = 6 - 3\sqrt{3} \approx 0.80, \quad \theta_2 = 3\sqrt{3} \approx 5.20$$

The corresponding trajectory of the point P when T changes from ∞ to 0 is shown in Fig. 1.

4. FEEDBACK OPTIMAL CONTROL

In order to construct a feedback optimal control it suffices to find the switching surfaces in the phase space xyz on which the sign of the control $u = \pm 1$ changes. On these surfaces, the length of one of the segments of constancy vanishes, that is, $\theta_1 = 0$ or $\theta_2 = 0$. From (3.4), we have here $\lambda = 0$ or $\lambda = 1$. According to (3.5), values of X and Y equal to $\pm 1/3$ and $\mp 1/2$ correspond to these values of λ respectively. From (3.5), we obtain the conditions

$$\zeta(\xi s^{-3} - s^{-1}/2) = \pm\sigma/3, \quad \zeta(\eta s^{-2} + s^{-1}) = \mp\sigma/2 \tag{4.1}$$

which are satisfied in the $\xi\eta$ plane on the switching curves when $z_0 \neq 0$. However, relations (4.1) are insufficient for determining the switching curves: for this, we require a direct analysis of relations (3.5) which will be carried out below.

Note that, in the feedback control, the initial conditions x_0, y_0, z_0 can be treated as the current values of the phase coordinates x, y, z. We consider relations (3.4) as formulae for the change of variables

$$\xi = z^{-3}x, \quad \eta = z^{-1}|z|^{-1}y, \quad \zeta = \operatorname{sign} z \tag{4.2}$$

in phase space. This change of variables, which introduces the self-similar variables ξ and η, enables one, when $z \neq 0$, to reduce the dimensions of the phase space by one and to construct the feedback optimal control in the $\xi\eta$ plane.

We will first consider the case when $z = 0$ separately. By analogy with (4.1), we obtain the conditions

$$xT^{-3} = \pm\sigma/3, \quad yT^{-2} = \mp\sigma/2 \tag{4.3}$$

from (3.6). These conditions are satisfied at the intersection of the switching surfaces with the $z = 0$ plane. When $z = 0$, conditions (4.3) define two halves of the semicubic parabolae which form the switching curve (SC) in the $z = 0$ plane, described by the equation

$$\gamma(x, y) \equiv 3x + 2y|y|^{3/2} = 0 \tag{4.4}$$

An analysis of the signs of σ on the branches of the SC (4.4) enables one to determine the signs of the controls on the different sides of the switching curve. As a result, we obtain the feedback optimal control when $z = 0$ in the form

$$\begin{aligned} &u(x, y, 0) = -\operatorname{sign}\gamma(x, y) \ \text{ when } \gamma \neq 0 \\ &u(x, y, 0) = \operatorname{sign} x = -\operatorname{sign} y \ \text{ when } \gamma = 0 \end{aligned} \tag{4.5}$$

When $z \neq 0$, the change of variables (4.2) transforms the first two equations of (1.3) to the form

$$\dot{\xi} = |z|^{-1}(\eta - 3u\zeta\xi), \quad \dot{\eta} = |z|^{-1}(1 - 2u\zeta\eta) \tag{4.6}$$

On dividing the first equation of (4.6) by the second, we obtain the linear equation in ξ

$$\frac{d\xi}{d\eta}=\frac{\eta-3\alpha\xi}{1-2\alpha\eta},\quad \alpha=u\zeta=\pm1 \tag{4.7}$$

The parameter α retains a constant value along the optimal trajectories which do not intersect the $z = 0$ plane. On integrating Eq. (4.7) in the case of constant α, we find its general solution

$$\xi=\Phi(\eta,\ \alpha,\ A)\equiv\alpha\eta-\tfrac{1}{3}+A|1-2\alpha\eta|^{3/2} \tag{4.8}$$

where A is an arbitrary constant. Note that the second equation of (4.6) enables one to determine the direction of motion along the optimal trajectories. If $\alpha = 1$, the motion occurs in the direction of an increase in η when $\eta < 1/2$ and in the direction of a decrease in η when $\eta > 1/2$. If, however, $\alpha = -1$, the motion occurs in the direction of a decrease in η when $\eta < -1/2$ and an increase in η when $\eta > -1/2$.

We will now construct the feedback optimal control. As was shown above, in order to do this it is sufficient to establish the sign of the control $u = \sigma$ at the initial instant of time $t = 0$ as a function of the initial data x_0, y_0, z_0. On changing to self-similar variables and returning to relations (3.5), the feedback problem can be formulated as follows: it is required to find the value of $\sigma = \pm1$ which corresponds to the solution of relations (3.5) (for fixed ξ, η, ζ, where $\zeta = \pm1$) with the least s, where $s > 0$ and $\lambda \in [0, 1]$.

We will now briefly describe the solution algorithm, and subsequently explain its most important features.

First, we note that relations (3.5) retain their form when ζ and σ change signs simultaneously. Consequently, when ζ is replaced by $-\zeta$, the required quantity σ also changes sign. It is therefore sufficient to construct the solution in the case when $\zeta = 1$ for arbitrary ξ and η and, in the case when $\zeta = -1$, simply to change the sign in the resulting dependence $\sigma(\xi, \eta)$.

Without loss of generality, we therefore put $\zeta = 1$ and eliminate λ using the second of Eqs (2.5). We obtain

$$\lambda=[\tfrac{1}{2}+\sigma(\eta s^{-2}+s^{-1})]^{1/2},\quad \sigma=\pm1 \tag{4.9}$$

Since $\lambda \in [0, 1]$, then, for fixed $\sigma = \pm1$ and η, the ranges of variation of s in which λ is real and $\lambda \leq 1$ are determined from (4.9). We substitute λ from (4.9) into the first equation of (3.5) and find the dependence of ξ on s, η and $\sigma = \pm1$. For fixed η, we shall denote these dependences by $\xi^+(s)$ and $\xi^-(s)$ for $\sigma = \pm1$. Subject to the condition $\lambda \in [0, 1]$, they define two curves in the s, ξ plane, each of which consists, generally speaking, of a finite number of arcs. We investigate these curves and find their domains of definition and the extrema in the whole range of variation in the argument s and the parameter η after which we analyse their arrangement with respect to one another. A line ξ = const is then mentally drawn in the $s\,\xi$ plane and the minimum value of the abscissa $s > 0$ is found for which this line intersects one of the above-mentioned curves. The value of $\sigma = \pm1$ which corresponds to that curve with which this intersection takes place determines the required control $u = \sigma$ for the data ξ, η and $\zeta = 1$ and the value of s which corresponds to this point of intersection is equal to the normalized optimal time: $s = T|z|^{-1}$ (of a normalized Bellman function).

We will now describe these operations in greater detail, taking account of the fact that all of the following constructions are only true when $s > 0$. From (3.5), we have

$$\xi^{\pm}(s)=\mp s^3/6-s^2/2-s\eta\pm(s^2/2\pm\eta\pm s)^{3/2}/3 \tag{4.10}$$

If $s \to +\infty$, then $\xi^{\pm}(s) \approx \pm(-1 + 1/\sqrt{2})s^3/6 \to \mp\infty$.

We now consider the function $\xi^+(s)$. In the case when $\sigma = 1$, the condition $\lambda \leq 1$ selects the set $s \in (0, s_2] \cup [s_1, +\infty)$, where $s_{1,2} = 1 \pm \sqrt{(1 + 2\eta)}$. The expression for $\xi^{\pm}(s)$ is determined if $s \in [s_5, +\infty)$, where $s_5 = -1 + \sqrt{(1 - 2\eta)}$. The derivative $d\xi^+/ds$ vanishes at the point s_5 and $s_7 = -1+\sqrt{(2(1 - 2\eta))}$ if s_5 and s_7 exist and $s_5 \leq s_7$. Furthermore, $d^2\xi^+/ds^2 < 0$ when $s = s_7$, that is, s_7 is a maximum point. It can be shown that, if s_1, s_2, s_5 and s_7 exist, then $s_2 \leq s_7 \leq s_1$, and $s_2 \geq s_5$.

If $\eta \geq 0$ then $s_2 \leq 0$ and $d\xi^+/ds < 0$ when $s \geq s_1$, that is, the function $\xi^+(s)$ is defined when $s \in [s_1, +\infty)$ and decreases from $\xi^+(s_1)$ to $-\infty$. If $-1/2 \leq \eta < 0$, then $s_5 > 0$, that is, the function $\xi^+(s)$ is defined when $s \in [s_5, s_2] \cup [s_1, +\infty)$. It has a null derivative when $s = s_5$, increases in the interval $[s_5, s_2]$ and decreases from $\xi^+(s_1)$ to $-\infty$ when $s \in [s_1, +\infty)$. If $\eta < -1/2$, then $s_5 > 0$ and the value of s_2 is undefined. Then, the function $\xi^+(s)$ is defined when $s \in [s_5, +\infty)$, $d\xi^+/ds = 0$ when $s = s_5$ and $\xi^+(s)$ increases up to a maximum at the point $s = s_7$ after which it decreases from $\xi^+(s_7)$ to $-\infty$.

We now consider the function $\xi^-(s)$. We require that $\lambda \leqslant 1$ in (4.9) and obtain that $s \in [s_5, +\infty)$ where $s_5 = -1 + \sqrt{(1-2\eta)}$. If $s \in (0, s_2] \cup [s_1, +\infty)$, then the function $\xi^-(s)$ from (4.10) is defined. Its derivative vanishes at the points s_1, s_2 and $s_3 = 1 - \sqrt{(2(1+2\eta))}$ if they exist and $s_3 \geqslant s_1$. In addition, $d^2\xi^- ds^2 > 0$ when $s_1 = s_3$, that is, $s = s_3$ is the point of a minimum.

If $\eta \geqslant 0$ then $s_2 \leqslant 0$ and the function $\xi^-(s)$ is defined when $s \in [s_1, +\infty)$ and $d\xi^-/ds = 0$ when $s = s_1$. The function $\xi^-(s)$ decreases from $\xi^-(s_1)$ up to the point of the minimum $s = s_3$ after which it increases from $\xi^-(s_3)$ up to $+\infty$. If $-1/2 \leqslant \eta < 0$ then $s_5 > 0$, that is, the dependence $\xi^-(s)$ is defined when $s \in [s_5, s_2] \cup [s_1, +\infty)$. The function $\xi^-(s)$ increases as s varies from $s = s_5$ to $s = s_2$, where $d\xi^-/ds = 0$ when $s = s_2$ and increases when $s \in [s_1, s_3]$ where $d\xi^-/ds = 0$ when $s = s_1$ and $s = s_3$. Then, $\xi^-(s)$ increases from $\xi^-(s_3)$ to $+\infty$. If $\eta < -1/2$, then $s_5 > 0$, the values of s_1, s_2 and s_3 are not defined and $d\xi^-/ds > 0$ when $s \geqslant s_5$, that is, the function $\xi^-(s)$ is defined when $s \in [s_5, +\infty)$ and increases over the whole of this interval up to $+\infty$.

We now make two remarks concerning the mutual arrangement of the pair of curves (one each from the two families investigated) for the same value of the parameter η. First, we find the point of intersection of the curves $\xi^+(s)$ and $\xi^-(s)$ which requires solutions of the equation

$$1-(\tfrac{1}{2}-\eta/s^2-1/s)^{3/2}=(\tfrac{1}{2}+\eta/s^2+1/s)^{3/2} \tag{4.11}$$

We square both sides of Eq. (4.11), reduce similar terms and then again square both sides of the equation and obtain an equation in s

$$((\eta/s^2+1/s)^2+2)((\eta/s^2+1/s)^2-\tfrac{1}{4})^2=0 \tag{4.12}$$

An analysis of the rods of Eq. (4.12) shows that only s_1, s_2 and s_5 are roots of Eq. (4.11) and, moreover, they are positive for just a single value of η. We shall denote coincident values $\xi^+ = \xi^-$ at the above-mentioned points by $\xi^{\pm}$.

Secondly, we establish that $\xi^{\pm}(s_5) > \xi^{\pm}(s_1)$ when and only when $-\sqrt{(3)}/4 < \eta \leqslant 0$.

As a result, it turns out to be convenient to pick out the four ranges of values of the parameter η which correspond to different mutual arrangements of the curves $\xi^+(s)$ and $\xi^-(s)$ which also determine the required control for all ξ and η with the exception of $\xi^{\pm}(s_1)$, $\xi^{\pm}(s_2)$ and $\xi^{\pm}(s_5)$.

When $\eta \geqslant 0$ for any $\xi < \xi^{\pm}(s_1)$, the minimum permissible abscissa s is reached on the curve $\xi^+(s)$. When $\xi > \xi^{\pm}(s_1)$, the same result holds for $\xi^-(s)$.

When $-\sqrt{(3)}/4 < \eta < 0$, the closed isolated curve for $s_5 \leqslant s \leqslant s_2$ is added to the curves $\xi^+(s)$ and the $\xi^-(s)$, which have the same characteristic singularities, where the curve $\xi^-(s)$ lies above the curve $\xi^+(s)$ and $\xi^{\pm}(s_5) < \xi^{\pm}(s_2)$. Moreover, $\xi^{\pm}(s_1) < \xi^{\pm}(s_5)$, that is, $\xi^{\pm}(s_1)$ lies below the lowest point of the closed isolated curve. Consequently, the required control is defined in the same way as in the preceding case.

When $-1/2 \leqslant \eta \leqslant -\sqrt{(3)}/4$, the inequality $\xi(s_1) > \xi(s_5)$ is satisfied and, for any $\xi < \xi^{\pm}(s_5)$, the minimum permissible abscissa s is attained on the curve $\xi^+(s)$. When $\xi > \xi^{\pm}(s_5)$, the same assertion holds for $\xi^-(s)$.

The close isolated curve disappears when $\eta < -1/2$ and the required control is specified as in the preceding case.

We now determine the control on the curves $\xi^{\pm}(s_1(\eta))$, $\xi^{\pm}(s_2(\eta))$ and $\xi^{\pm}(s_5(\eta))$ in the ξ, η plane. We recall that the dependences of s_1, s_2 and s_5 on η have been presented above. By (4.9), we have $\lambda = 0$ and $\sigma = -1$ on the curve $\xi^{\pm}(s_1(\eta))$, that is, the time interval in which it is necessary to take $u = 1$ is equal to zero. Consequently, it is necessary to take $u = -1$ on the curve $\xi^{\pm}(s_1(\eta))$ and it is a switching curve when $\eta > -\sqrt{(3)}/4$. Similarly, on the curve $\xi^{\pm}(s_2(\eta))$, one must use $u = -1$ when $-1/2 \leqslant \eta < 0$ but this curve will not be a switching curve. It is easy to show, using the same method, that we have $u = 1$ when $\eta < 0$ on the curve $\xi^{\pm}(s_5(\eta))$. This curve serves as a switching curve.

We now completely present the feedback optimal control. To be specific, we shall take $\zeta > 0$ and $\zeta = 1$. The switching curve in the $\xi\eta$ plane is defined by the equalities

$$\xi = f(\eta) = \begin{cases} \Phi(\eta,\ 1,\ \tfrac{1}{3}), & \eta \leqslant \eta^* \\ \Phi(\eta,\ -1,\ -\tfrac{1}{3}), & \eta > \eta^*;\ \eta^* = -\sqrt{3}/4 \end{cases} \tag{4.13}$$

where the notation of (4.8) is used. The switching curve is continuous and has a kink at the point K with the coordinates $\xi^* = 1/12$, $\eta^* = -\sqrt{(3)}/4$. This curve is represented by the solid line in Figs 2 and 3. On account of the fact that the scale in Fig. 3 is smaller than that in Fig. 2, the points K and R shown in Fig. 2 are practically indistinguishable in Fig. 3 and are therefore not labelled. On the other hand, the scale used in Fig. 3 enables us to depict all the characteristic phase trajectories, the important part of which is missing in Fig. 2. The rest of the notation employed in Figs 2 and 3 is identical. To be specific, we shall henceforth mainly refer to Fig. 2. The branches of the switching curve corresponding to $\eta < \eta^*$ and $\eta > \eta^*$ are denoted by the letters M and N respectively. In the $\xi\eta$ plane, we have

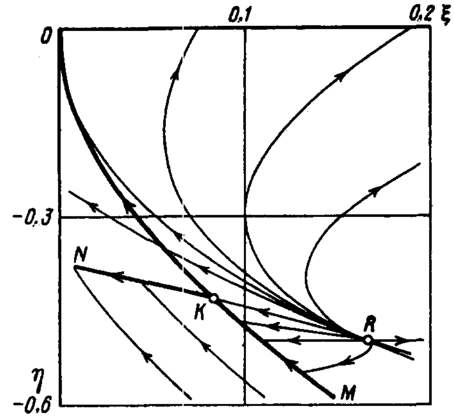

Fig. 2.

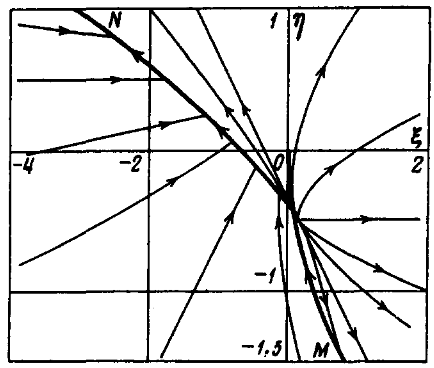

Fig. 3.

$$u = 1 \text{ when } \xi < f(\eta)$$

$$u = 1 \text{ when } \xi = \Phi(\eta, 1, 1/3), \eta \leqslant 0 \tag{4.14}$$

$$u = -1 \text{ at the remaining points of the } \xi\eta \text{ plane}$$

Hence $u = 1$ to the left of and below the switching curve (4.13), on its segment KM to the right of and below point K and also on the arc of the curve $\xi = \Phi(\eta, 1, 1/3)$ which joins the origin of the system of coordinates and the point K (see Fig. 2), where this arc is a part of the switching curve. In the remaining part of the $\xi\eta$ plane, we have $u = -1$.

When $z < 0$, $\zeta = -1$, the switching curve remains the same and one simply has to interchange the positions of the set of points $\xi\eta$ where $u = 1$ and $u = -1$ in relations (4.14). So, the synthesis of the optimal control $u(x, y, z)$ is completely determined by relations (4.2), (4.4), (4.5), (4.8), (4.13) and (4.14) for all x, y, z.

We now describe the set of optimal trajectories which, in the variables ξ and η, consist of arcs of the curves (4.8). Suppose that the initial point x, y, z is specified and, to be specific, we shall assume that $z > 0$. According to formulae (3.4), we find that ξ, η and $\zeta = 1$.

If a point $\xi\eta$ lies on the curve $\xi = \Phi(\eta, 1, 1/3)$, where $\eta \leqslant 0$, then motion occurs along this curve $MK0$ with a control $u = 1$ until it reaches the origin of the system of coordinates.

All the remaining optimal trajectories also arrive at the origin of the system of coordinates along this curve. An exception is the segment $R0$ of the curve $\xi = \Phi(\eta, -1, 1/3)$ when $\eta \in [-1/2, 0]$: this segment is a phase trajectory for $u = -1$ which begins at the point R with the coordinates $(1/6, -1/2)$ and reaches the origin of the system of coordinates. Phase trajectories are denote by thin lines in Figs 2 and 3 and arrows indicate the direction of the motion.

If the initial point lies on the curvilinear arc

$$\eta \leqslant 0, \quad \Phi(\eta, 1, 1/3) < \xi < \Phi(\eta, -1, 1/3) \tag{4.15}$$

then the optimal trajectory consists of the segment with $u = -1$ until it reaches the curve $\xi = \Phi(\eta, 1, 1/3)$ and of the subsequent motion along this curve with $u = 1$.

If the initial point lies in the domain $\xi < f(\eta)$, the motion initially occurs with $u = 1$ until it intersects the curve $\Phi = \xi(\eta, -1, -1/3)$ which is the part KN of the switching curve (4.13) (see Fig. 2) and then with $u = -1$ along this curve which departs to infinity. By (3.4), we have $z = 0$ at an infinitely distant point of the $\xi\eta$ plane. At infinity, z changes sign and, then, $z < 0$, $\zeta = -1$. The phase trajectory continues, arriving, when $u = -1$, from infinity along the curve $\xi = \Phi(\eta, 1, 1/3)$ and arrives along this curve at the origin of the system of coordinates. Note that motion through an infinitely distant point occurs without a change in the control and takes a finite time.

It remains to consider initial points in the domain $\xi > f(\eta)$ but outside of the curvilinear corner (4.15). Here, we initially have $u = -1$ and the trajectory $\xi = \Phi(\eta, -1, A)$ departs to infinity, where $A > -1/3$. When $\zeta = -1$ and $u = -1$, motion subsequently occurs along the curves $\xi = \Phi(\eta, 1, -A)$ with a change in the sign of A. These curves lie in the domain $\xi < f(\eta)$ and persist in the branch KN of the switching curve $\xi = \Phi(\eta, -1, -1/3)$. A trajectory with $u = 1$ departs to infinity along this curve, where the sign of z changes again. Later, when $\zeta = 1$, motion occurs, when $u = 1$, along the curve $\xi = \Phi(\eta, 1, 1/3)$ until it reaches the origin of the system of coordinates.

Note that certain phase trajectories contain segments of the lines $\xi = \pm\eta - 1/3$ and $\eta = \pm(2\alpha)^{-1}$, which correspond to the values $A = 0$ and $A = \infty$ in (4.8) respectively. On departing to infinity along these lines the variable x (in the case of the straight line with $A = 0$) or the variable y (in the case of the straight line with $A = \infty$) simultaneously vanishes together with z, as is easily shown using (3.4). In other respects these lines are treated in the same manner as the remaining trajectories (4.8).

Hence for any initial point x, y, z, the motion is completely described by the trajectories of Figs 2 and 3 and contains not more than two segments where the control is constant. In this case, the sign of z cannot change more than twice.

We now present the results of an investigation of the normalized optimal time s as a function of ξ and η. The dependence of s on ξ for different fixed values of η is studied where s_1, s_2, s_5 are again considered to be the functions of η which were introduced above. When $\eta \geq 0$, the function $s(\xi, \eta)$ decreases as ξ increases if $\xi < \xi^{\pm}(s_1)$ and has a discontinuity if $\xi = \xi^{\pm}(s_1)$. It increases on passing from $\xi < \xi^{\pm}(s_1)$ to $\xi > \xi^{\pm}(s_1)$ and when ξ increases from $\xi = \xi^{\pm}(s_1)$ to $+\infty$.

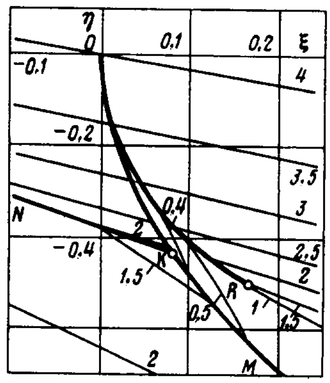

Fig. 4.

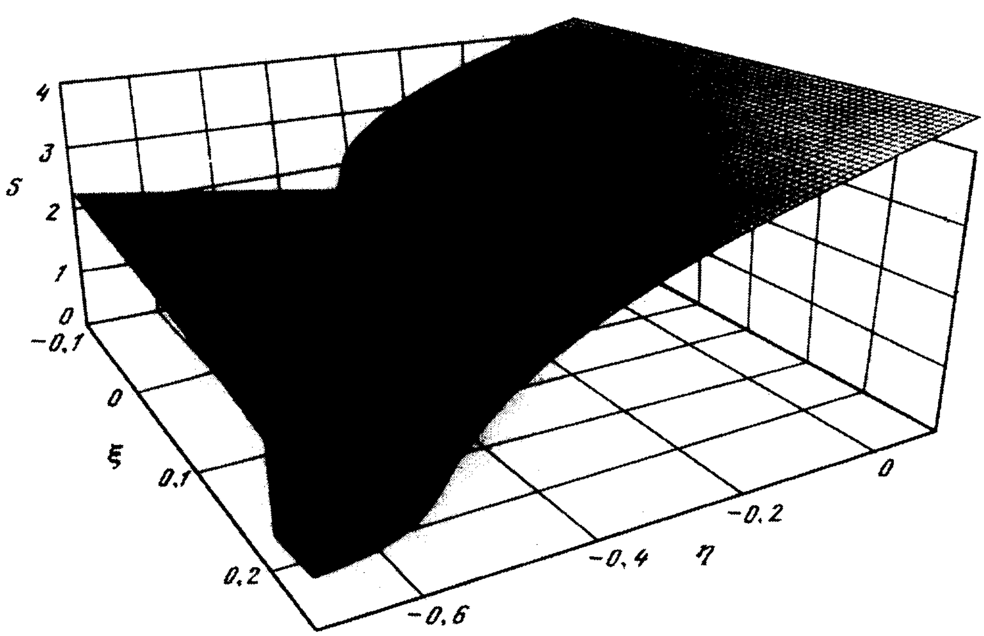

Fig. 5.

When $-\sqrt{(3)}/4 < \eta < 0$, the function $s(\xi, \eta)$ decreases as ξ increases if $\xi < \xi^{\pm}(s_1)$ and has a discontinuity if $\xi = \xi^{\pm}(s_1)$. It increases on passing from $\xi < \xi^{\pm}(s_1)$ to $\xi > \xi^{\pm}(s_1)$. There is a further discontinuity when $\xi = \xi^{\pm}(s_5)$. The function $s(\xi, \eta)$ decreases on passing from $\xi < \xi^{\pm}(s_5)$ to $\xi > \xi^{\pm}(s_5)$ but increases when $\xi^{\pm}(s_5) \leqslant \xi \leqslant \xi^{\pm}(s_2)$. There is a further discontinuity when $\xi = \xi^{\pm}(s_2)$. The function $s(\xi, \eta)$ also increases on passing from $\xi < \xi^{\pm}(s_2)$ to $\xi > \xi^{\pm}(s_2)$ and when ξ increases from $\xi = \xi^{\pm}(s_2)$ to $+\infty$.

When $-1/2 \leqslant \eta \leqslant -\sqrt{(3)}/4$, the function $s(\xi, \eta)$ decreases as ξ increases if $\xi < \xi^{\pm}(s_5)$ and has a discontinuity if $\xi = \xi^{\pm}(s_5)$. It decreases on passing from $\xi < \xi^{\pm}(s_5)$ to $\xi > \xi^{\pm}(s_5)$ but it increases when $\xi^{\pm}(s_5) \leqslant \xi \leqslant \xi^{\pm}(s_2)$. The next discontinuity occurs when $\xi = \xi^{\pm}(s_2)$. The function $s(\xi, \eta)$ increases on passing from $\xi < \xi^{\pm}(s_2)$ to $\xi > \xi^{\pm}(s_2)$ and when ξ increases from $\xi = \xi^{\pm}(s_2)$ to $+\infty$.

When $\eta < -1/2$, the function $s(\xi, \eta)$ decreases as ξ increases when $\xi < \xi^{\pm}(s_5)$ and has a discontinuity if $\xi = \xi^{\pm}(s_5)$. It decreases on passing from $\xi < \xi^{\pm}(s_5)$ to $\xi > \xi^{\pm}(s_5)$ and increases as ξ increases from $\xi = \xi^{\pm}(s_5)$ to $+\infty$.

In Fig. 4, the thin lines are level lines of the function $s(\xi, \eta)$ and the bold lines are the lines of discontinuity of this function. The rest of the notation is the same as in Fig. 3. A three-dimensional graph of the function $s(\xi, \eta)$ is shown in Fig. 5 where the darker the background, the smaller the corresponding value.

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